

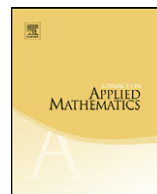


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## A Franklin type involution for squares

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## ABSTRACT

We find an involution as a combinatorial proof of Ramanujan's partial theta identity. Based on this involution, we obtain a Franklin type involution on the set of partitions into distinct parts with the smallest part being odd. Compared with the involution of Bessenrodt and Pak, our involution possesses a weight-preserving property that leads to a combinatorial proof of a weighted partition theorem derived by Alladi from Ramanujan's partial theta identity. This gives an indirect answer to a question of Berndt, Kim and Yee. Moreover, we obtain a partition theorem based on Andrews' identity and provide a combinatorial proof via certain weight assignment for our involution. A specialization of this partition theorem is related to an identity of Andrews concerning partitions into distinct nonnegative parts with the smallest part being even. Finally, we give an extension of our partition theorem which corresponds to a generalization of Andrews' identity.

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## 1. Introduction

The main result of this paper is a Franklin type involution for squares which is related to Ramanujan's partial theta identity and an identity of Andrews. As applications of this involution, we give an indirect solution to a problem proposed by Berndt, Kim and Yee [8] by providing a combinatorial interpretation of a partition theorem derived by Alladi [1] from Ramanujan's partial theta identity. Furthermore, we obtain a partition theorem based on Andrews' identity. A specialization of this theorem is related to an identity of Andrews on partitions into distinct nonnegative parts with the smallest part being even. Finally, we find a more general form of our partition theorem which corresponds to a generalization of Andrews' identity.

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Recall that the celebrated involution of Franklin gives a combinatorial interpretation of Euler's pentagonal number theorem as stated below

$$(q; q)_{\infty} = 1 + \sum_{k=1}^{\infty} (-1)^k (q^{k(3k-1)/2} + q^{k(3k+1)/2}), \quad (1.1)$$

where the  $q$ -shifted factorial is defined by

$$(a; q)_n := (1-a)(1-aq) \cdots (1-aq^{n-1}), \quad n \geq 1,$$

and

$$(a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n, \quad |q| < 1.$$

Let  $D$  denote the set of integer partitions into distinct parts, and let  $D(n)$  denote the set of partitions of  $n$  into distinct parts. The relation (1.1) can be reinterpreted as the following number-theoretic identity

$$\sum_{\lambda \in D(n)} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^k, & \text{if } n = k(3k \pm 1)/2, \\ 0, & \text{otherwise,} \end{cases} \quad (1.2)$$

where  $\ell(\lambda)$  denotes the number of parts of  $\lambda$ .

Our Franklin type involution for squares will be concerned with the set of partitions of a nonnegative integer into distinct parts with the smallest part being odd. Let us use  $P_{do}$  to denote the set of such partitions, and use  $P_{do}(n)$  to denote the set of such partitions of  $n$ . To be more specific, we obtain the following number-theoretic identity which is analogous to (1.2),

$$\sum_{\lambda \in P_{do}(n)} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \quad (1.3)$$

It is clear that (1.3) implies a theorem of Fine [9] concerning the parity of the number of partitions in  $P_{do}(n)$ . Bessenrodt and Pak [7] constructed an involution for Fine's theorem, and Yee [15] gave an indirect bijective proof.

Moreover, for various weight assignments  $\omega(\lambda)$  to partitions  $\lambda \in P_{do}$ , our involution turns out to be sign-reversing and weight-preserving. This property leads to several number-theoretic identities of the following form:

$$\sum_{\lambda \in P_{do}(n)} \omega(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \quad (1.4)$$

The first case is related to a problem proposed by Berndt, Kim and Yee [8] concerning a combinatorial interpretation of a number-theoretic identity derived by Alladi [1] from the partial theta identity from Ramanujan's lost notebook, see [6, Entry 1.6.2] or [13, p. 28],

$$1 + \sum_{k=1}^{\infty} \frac{(-q; q)_{k-1} (-a)^k q^{k(k+1)/2}}{(aq^2; q^2)_k} = \sum_{k=0}^{\infty} (-a)^k q^{k^2}. \quad (1.5)$$

Defining a weight function in terms of the gaps between the parts of partitions in  $P_{do}$ , Alladi [1] derived a partition theorem in the above form, see, Section 4. Though Berndt, Kim and Yee [8] have

found a bijective proof of Ramanujan's identity (1.5), it is not clear whether their involution implies a combinatorial interpretation of Alladi's weighted partition theorem. As will be seen, our Franklin type involution gives a combinatorial proof of Alladi's weighted partition theorem, whereas the involution of Bessenrodt and Pak [7] is not weight-preserving as far as Alladi's theorem is concerned.

The second case is concerned with a weighted partition theorem obtained by Alladi [2] from the following partial theta identity of Andrews [4, p. 157],

$$\sum_{n=0}^{\infty} q^{2n} (q^{2n+2}; q^2)_{\infty} (aq^{2n+1}; q^2)_{\infty} = \sum_{k=0}^{\infty} (-a)^k q^{k^2}. \quad (1.6)$$

By giving a weight function in terms of the odd parts of partitions in  $P_{do}$ , Alladi [2] derived a partition theorem in the form of (1.4). It turns out that our involution also applies to this partition theorem in terms of a different weight assignment.

As the third application of our involution, we give a combinatorial proof of a number-theoretic theorem on partitions into distinct parts with smallest part being even derived from Andrews' identity (1.6). Moreover, we note that a special case of this partition theorem is related to an identity of Andrews, first proposed as a problem in [3]. A generating function proof was given by Stenger [14].

To conclude this paper, we extend our involution to derive a more general identity

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2mn} (q^{2mn+2m}; q^{2m})_{\infty} (aq^{2mn+1}; q^2)_{\infty} \\ &= 1 + \sum_{k=1}^{\infty} (-a)^k q^{k^2} \prod_{j=1}^k (1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}), \end{aligned} \quad (1.7)$$

which reduces to the following identity of Andrews [4, p. 157] when setting  $a = -1$ ,

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2mn} (q^{2mn+2m}; q^{2m})_{\infty} (-q^{2mn+1}; q^2)_{\infty} \\ &= 1 + \sum_{k=1}^{\infty} q^{k^2} \prod_{j=1}^k (1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}). \end{aligned} \quad (1.8)$$

Notice that (1.8) is a generalization of (1.6).

## 2. An involution for Ramanujan's identity

In this section, we shall construct an involution which leads to a combinatorial proof of Ramanujan's partial theta identity as stated in the previous section:

$$1 + \sum_{k=1}^{\infty} \frac{(-q; q)_{k-1} (-a)^k q^{k(k+1)/2}}{(aq^2; q^2)_k} = \sum_{k=0}^{\infty} (-a)^k q^{k^2}. \quad (2.1)$$

This involution plays a key role in the Franklin type involution presented in the next section which can be viewed as a bijective proof of Alladi's partition theorem derived from (2.1) with respect to certain weight assignment.

Berndt, Kim and Yee [8] provided a bijective proof of (2.1) based on the interpretation of the numerator  $(-q; q)_{k-1} q^{k(k+1)/2}$  in terms of parity sequences. We find a bijective proof of (2.1) which leads

to a combinatorial proof of Alladi's partition theorem derived from identity (2.1). Yee [16] independently obtains a bijective proof of (2.1) in order to answer a question of Andrews [5] concerning the upper even parity index and the upper odd parity index of partitions.

Let  $D_k$  be the set of partitions  $\pi$  into  $k$  distinct parts with the smallest part being 1 such that  $\pi_i - \pi_{i+1} \leq 2$ , and let  $E_k$  denote the set of partitions  $\sigma$  with even parts not exceeding  $2k$ , that is, each  $\sigma_i$  is even and  $\sigma_1 \leq 2k$ . We are going to establish an involution on  $D_k \times E_k$ . Throughout this paper,  $T_k$  stands for the triangular partition  $(2k - 1, 2k - 3, \dots, 3, 1)$ .

**Theorem 2.1.** *There exists an involution on the set  $D_k \times E_k$  under which the pair of partitions  $(T_k, \emptyset)$  remains invariant.*

To construct the desired involution on  $D_k \times E_k$ , we introduce a statistic called the modular leg hook length of a partition in  $D_k$ . Adopting the notation in [12], we use  $[\lambda]_2$  to denote the 2-modular diagram of a partition  $\lambda$  defined to be a Young diagram filled with 1 or 2 such that the last cell of row  $i$  is filled with 1 if and only if  $\lambda_i$  is odd. Given a partition  $\pi = (\pi_1, \pi_2, \dots, \pi_k) \in D_k$ , let us consider the 2-modular diagram. Suppose that  $\pi_i$  is an even part other than the largest part, we can associate it with a modular leg hook  $H_i$  which consists of the squares in the  $i$ -th row in the 2-modular diagram and the squares in first column above the  $i$ -th row. For a modular leg hook  $H_i$ , the length of this hook, denoted by  $|H_i|$ , is defined to be the sum of the numbers filled in the hook, and its height is referred to the number of squares in the first column.

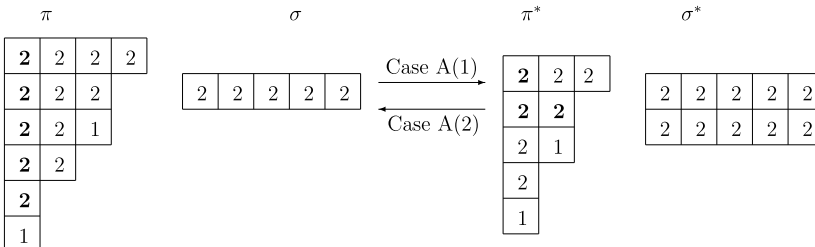
We are now ready to describe the construction of the involution on  $D_k \times E_k$ . Let us denote this involution by  $\varphi$ .

*The involution  $\varphi$  on  $D_k \times E_k$ :* Given a pair of partitions  $(\pi, \sigma) \in D_k \times E_k$ , represent  $\pi$  and  $\sigma$  by their 2-modular diagrams, respectively. In fact, the desired involution consists of two involutions.

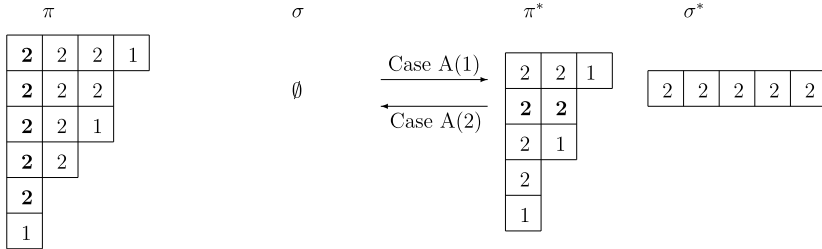
*Part A:* We have the following two cases.

- (1) Suppose that there exists a modular leg hook in  $\pi$  such that after the deletion of this hook the resulting partition is in  $D_{k-1}$ , then we choose such a hook with maximum height and denote it by  $H_r(\pi)$ . If  $|H_r(\pi)| \geq \sigma_1$ . Then delete  $H_r(\pi)$  from  $\pi$  and add it to  $\sigma$  as a new part. We denote the resulting partitions by  $\pi^*$  and  $\sigma^*$ , respectively. Since  $|H_r(\pi)| \leq 2k - 2$ , we have  $(\pi^*, \sigma^*) \in D_{k-1} \times E_{k-1}$ .
- (2) Suppose that either there is the modular leg hook  $H_r(\pi)$  in  $\pi$  and  $|H_r(\pi)| < \sigma_1$  or there does not exist the modular leg hook  $H_r(\pi)$  in  $\pi$  and  $\pi_1 + 2 < \sigma_1$ . Then insert  $\sigma_1$  into  $\pi$  as a modular leg hook in the 2-modular diagram of  $\pi$ . To be precise, this operation can be described as follows. Let  $i$  be the largest positive integer such that  $\sigma_1 - 2i > \pi_{i+1}$ , that is, for  $j > i$  we have  $\sigma_1 - 2j < \pi_{j+1}$ . Then we add 2 to each of the first  $i$  parts  $\pi_1, \pi_2, \dots, \pi_i$ , and insert  $\sigma_1 - 2i$  as a new part before the part  $\pi_{i+1}$ . Since  $\sigma_1 \leq 2k - 2$  and any two consecutive parts of  $\pi$  differ by at most 2, the resulting pair of partitions, denoted by  $(\pi^*, \sigma^*)$ , belongs to  $D_{k+1} \times E_k$ . Furthermore, there exists the modular leg hook  $H_r(\pi^*)$  in  $\pi^*$  and  $|H_r(\pi^*)| \geq \sigma_1^*$ .

Below is an example.



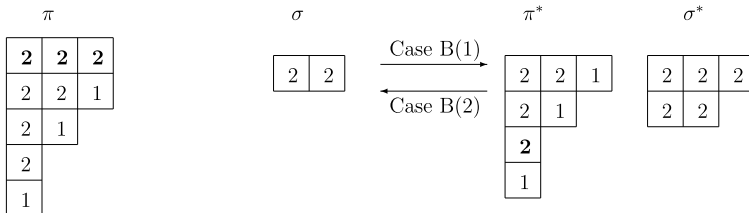
For the case when there does not exist a modular leg hook, we have the following example.



**Part B:** Suppose that there does not exist the modular leg hook  $H_r(\pi)$  in  $\pi$  and  $\sigma_1 \leq \pi_1 + 2$ . If  $\pi$  has even parts, then we choose the largest even part of  $\pi$ , and denote it by  $\pi_r$ . We consider the following two cases.

- (1) If  $\pi_r \geq \sigma_1$ , then remove the part  $\pi_r$  in  $\pi$  and add it to  $\sigma$ . We denote the resulting partitions by  $\pi^*$  and  $\sigma^*$ , respectively. Since  $\pi_r \leq 2k - 2$ , we see that  $(\pi^*, \sigma^*) \in D_{k-1} \times E_{k-1}$ .
- (2) If  $\pi_r < \sigma_1$ , then remove the part  $\sigma_1$  from  $\sigma$  and add it to  $\pi$ . Denote the resulting partitions by  $\pi^*$  and  $\sigma^*$ , respectively. Since  $\pi_i - \pi_{i+1} \leq 2$  for each  $i$ ,  $\sigma_1$  can be inserted either between two odd parts of  $\pi$  or at the top of  $\pi$ . Therefore,  $(\pi^*, \sigma^*) \in D_{k+1} \times E_k$ .

Here is an example.



In an extreme case, for  $\pi = (7, 6, 5, 4, 3, 2, 1)$  and  $\sigma = \emptyset$ , we have  $\pi^* = (7, 5, 4, 3, 2, 1)$  and  $\sigma^* = (6)$  under the involution  $\varphi$ .

Finally, we are left with the case when  $\pi$  has no even parts and  $\sigma$  is the empty partition. In this situation, there is only one pair of partitions  $(T_k, \emptyset)$ , which is defined as the fixed point of the involution.

It is straightforward to check that the above correspondence is an involution. Except for the fixed point, the mapping changes the number of even parts of  $\pi$  by 1 and preserves the number of odd parts at the same time. Indeed, the above involution serves as a combinatorial proof of Ramanujan's partial theta identity (2.1).

**Proof of (2.1).** Note that the generating function for partitions  $\pi \in D_k$  equals

$$(-q; q)_{k-1} q^{k(k+1)/2},$$

and the generating function for partitions  $\sigma \in E_k$  equals

$$\frac{1}{(q^2; q^2)_k}.$$

Thus the left hand side of (2.1) corresponds to partitions  $(\pi, \sigma) \in \bigcup_{k=0}^{\infty} D_k \times E_k$  with weight  $(-1)^{\ell(\pi)} a^{\ell(\pi)+\ell(\sigma)}$ . Notice that the involution  $\varphi$  changes the parity of  $\ell(\pi)$  and preserves the quantity  $\ell(\pi) + \ell(\sigma)$ . The fixed point  $(T_k, \emptyset)$  corresponds to the right hand side of (2.1). In view of the involution  $\varphi$ , we obtain the identity (2.1).  $\square$

### 3. A Franklin type involution for squares

In this section, we shall construct a Franklin type involution on  $P_{do}(n)$ , namely, the set of partitions of  $n$  into distinct parts with the smallest part being odd, where the involution  $\varphi$  on  $D_k \times E_k$  given in the previous section serves as the main ingredient. This involution will be used to give a combinatorial proof of a partition theorem derived by Alladi from Ramanujan's partial theta identity.

It should be noted that Bessenrodt and Pak [7] have established a different involution on  $P_{do}(n)$  by using Vahlen's involution and Sylvester's transformation, which leads to the following theorem in the spirit of Euler's pentagonal number theorem.

**Theorem 3.1.** *For any positive integer  $n$ , we have*

$$\sum_{\lambda \in P_{do}(n)} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.1 implies the following theorem of Fine [9], see also [7,11].

**Theorem 3.2.** *The number of partitions of  $n$  into distinct parts with the smallest being odd is odd if and only if  $n$  is a square.*

An indirect bijective proof of Fine's theorem has been given by Yee [15]. Compared with the involution of Bessenrodt and Pak, our involution also leads to the above theorem, and it further possesses a weight-preserving property for the purpose of giving a combinatorial proof of Alladi's theorem. Meanwhile, both the involution of Bessenrodt and Pak and our involution have weighted versions for another partition theorem of Alladi and for the partition theorems derived from identities of Andrews. Our involution, denoted by  $\Psi$ , can be described as follows.

**Step 1.** Extraction of parts from  $\lambda$ : For a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in P_{do}(n)$ , represent it by the 2-modular diagram  $[\lambda]_2$ , from which we can construct a pair of partitions  $(\pi, \sigma) \in D_k \times E_k$ . Initially, set  $\pi = \lambda$ ,  $\sigma = \emptyset$  and  $t = k$ . Then iterate the following procedure until  $t = 0$ :

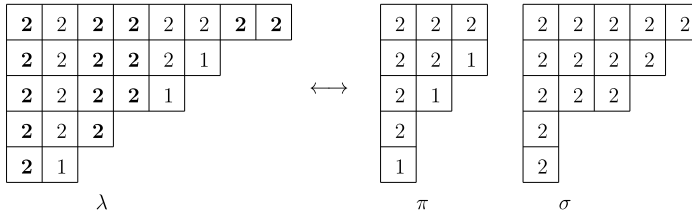
- Suppose that there exists  $i$  such that  $\pi_t - \pi_{t+1} = 2i + r_t$ , where  $1 \leq r_t \leq 2$  and  $\pi_{k+1}$  is defined to be 0.
- Subtract  $2i$  from each of the parts  $\pi_1, \pi_2, \dots, \pi_t$ .
- Rearrange the parts to form a new partition  $\pi$  and add  $i$  parts of size  $2t$  to  $\sigma$ . Replace  $t$  by  $t - 1$ .

When  $t = 0$ , we get a pair of partitions  $(\pi, \sigma) \in D_k \times E_k$ . It is clear that

$$\ell(\lambda) = \ell(\pi), \quad \ell_e(\lambda) = \ell_e(\pi), \quad \ell_o(\lambda) = \ell_o(\pi),$$

where  $\ell_e(\lambda)$  (resp.  $\ell_o(\lambda)$ ) denotes the number of even (resp. odd) parts of  $\lambda$ .

Here is an example.



**Step 2.** Apply the involution  $\varphi$  on  $D_k \times E_k$ : For a pair of partitions  $(\pi, \sigma) \in D_k \times E_k$ , use the involution  $\varphi$  to generate a pair of partitions  $(\pi^*, \sigma^*) \in D_{k+1} \times E_k$  or  $D_{k-1} \times E_{k-1}$ .

**Step 3.** Insertion of parts of  $\sigma^*$  to  $\pi^*$ : For a pair of partitions  $(\pi^*, \sigma^*) \in D_{k+1} \times E_k$  or  $D_{k-1} \times E_{k-1}$ , consider their 2-modular diagrams. Let  $\lambda^* = \pi^* + c_2(\sigma^*)$ , where  $c_2(\sigma^*)$  denotes the 2-modular conjugate partition obtained from  $[\sigma^*]_2$ , and for partitions  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $\mu = (\mu_1, \mu_2, \dots)$ ,  $\lambda + \mu$  is defined to be the partition  $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$ . Clearly, we have  $\lambda^* \in P_{do}(n)$ . It is obvious that

$$\ell(\lambda^*) = \ell(\pi^*), \quad \ell_e(\lambda^*) = \ell_e(\pi^*), \quad \ell_o(\lambda^*) = \ell_o(\pi^*).$$

Based on the above procedure, we see that the mapping  $\Psi$  is a bijection. Moreover, it is easily seen that

$$\ell(\lambda^*) = \ell(\lambda) \pm 1, \quad \ell_e(\lambda^*) = \ell_e(\lambda) \pm 1, \quad \ell_o(\lambda^*) = \ell_o(\lambda), \quad (3.1)$$

where the  $\pm$  sign means either plus or minus. In other words,  $\Psi$  changes the parity of the number of parts. It is easy to check that only when  $n$  is a square, say,  $n = k^2$ , there is exactly one partition which is undefined for  $\Psi$ , that is,  $\lambda = (2k - 1, 2k - 3, \dots, 3, 1)$ . Therefore, the involution  $\Psi$  gives a combinatorial proof of Theorem 3.1.

For example, when  $n = 10$ , there are six partitions in  $P_{do}(10)$ , namely,

$$\begin{aligned} &9 + 1, \quad 7 + 3, \quad 4 + 3 + 2 + 1, \\ &7 + 2 + 1, \quad 6 + 3 + 1, \quad 5 + 4 + 1. \end{aligned}$$

The involution  $\Psi$  gives the following correspondence

$$9 + 1 \leftrightarrow 7 + 2 + 1, \quad 7 + 3 \leftrightarrow 6 + 3 + 1, \quad 4 + 3 + 2 + 1 \leftrightarrow 5 + 4 + 1.$$

While under the involution of Bessenrodt and Pak [7], the corresponding relations are given by

$$9 + 1 \leftrightarrow 6 + 3 + 1, \quad 7 + 3 \leftrightarrow 7 + 2 + 1, \quad 4 + 3 + 2 + 1 \leftrightarrow 5 + 4 + 1.$$

#### 4. Alladi's partition theorems

In this section, we apply the involution  $\Psi$  presented in the previous section to give a combinatorial interpretation of a weighted partition theorem derived by Alladi [1] from Ramanujan's partial theta identity (1.5). While Berndt, Kim and Yee [8] constructed an involution for the identity (1.5), they raised the question of how to translate their involution into a combinatorial proof of Alladi's weighted partition theorem. Even though our involution is not a direct answer to their question, it is likely that

there is no easy way to make the translation. If so, our combinatorial interpretation can be considered as an indirect answer to the question of Berndt, Kim and Yee. The theorem of Alladi is stated as follows.

**Theorem 4.1.** For  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in P_{do}$ , define  $\delta_i$  to be the least integer  $\geq (\lambda_i - \lambda_{i+1})/2$ , where  $\lambda_{l+1}$  is defined to be 0. Define the weight of  $\lambda$  by

$$\omega_g(\lambda) = (-1)^l \prod_{i=1}^l a^{\delta_i}. \quad (4.1)$$

Then we have

$$\sum_{\lambda \in P_{do}(n)} \omega_g(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \quad (4.2)$$

**Proof.** For  $\lambda \in P_{do}(n)$ , let  $(\pi, \sigma)$  be the pair of partitions obtained from  $\lambda$  in Step 1 of the Franklin type involution  $\Psi$ . It can be seen that the exponent of  $a$  in  $\omega_g(\lambda)$  equals  $\ell(\pi) + \ell(\sigma)$ . It is also clear that the quantity  $\ell(\pi) + \ell(\sigma)$  remains unchanged in Step 2, that is

$$\ell(\pi) + \ell(\sigma) = \ell(\pi^*) + \ell(\sigma^*).$$

Let  $\lambda^* = \pi^* + c_2(\sigma^*)$  in Step 3, then the exponent of  $a$  in  $\omega_g(\lambda^*)$  equals  $\ell(\pi^*) + \ell(\sigma^*)$ . Thus the involution  $\Psi$  preserves the exponent of  $a$  in  $\omega_g(\lambda)$ . In view of (3.1), we see that  $\omega_g(\lambda)$  and  $\omega_g(\lambda^*)$  have opposite signs. Therefore, the partitions  $\lambda$  in  $P_{do}(n)$  cancel each other except for the partition  $\lambda = (2k-1, 2k-3, \dots, 3, 1)$  which has weight  $(-a)^k$  for  $n = k^2$ . This completes the proof.  $\square$

For example, when  $n = 9$ , there are five partitions in  $P_{do}(9)$ , that is,

$$\begin{array}{ccc} 8+1, & 6+3, \\ 9, & 6+2+1, & 5+3+1. \end{array}$$

Under the involution  $\Psi$ , the partitions are paired as follows

$$8+1 \leftrightarrow 9, \quad 6+3 \leftrightarrow 6+2+1,$$

while the triangular partition  $5+3+1$  remains fixed. Meanwhile, the weights of the partitions are given by

$$\omega_g(8+1) = a^5, \quad \omega_g(6+3) = a^4,$$

and

$$\omega_g(9) = -a^5, \quad \omega_g(6+2+1) = -a^4, \quad \omega_g(5+3+1) = -a^3.$$

In view of (3.1), it can be seen that the Franklin type involution  $\Psi$  preserves the number of odd parts of  $\lambda \in P_{do}(n)$ . Thus, the involution  $\Psi$  can be used to give a combinatorial interpretation of another weight partition theorem derived by Alladi [2] from Andrews' identity (1.6).



**Theorem 4.2.** For  $\lambda \in P_{do}$ , define the weight of  $\lambda$  by

$$\omega_o(\lambda) = (-1)^l a^{\ell_o(\lambda)}. \quad (4.3)$$

Then we have

$$\sum_{\lambda \in P_{do}(n)} \omega_o(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \quad (4.4)$$

**Proof.** Let  $\lambda \in P_{do}(n)$ . From (3.1), it is easily seen that the involution  $\psi$  changes the number of even parts of  $\lambda$  by 1 and preserves the number of odd parts. Consequently, the involution  $\psi$  preserves the exponent of  $a$  given in the weight  $\omega_o(\lambda)$  and reverses the sign of  $\omega_o(\lambda)$ . When  $n$  is a square, say,  $n = k^2$ , there exists exactly one partition which is undefined for  $\psi$ , that is  $\lambda = (2k-1, 2k-3, \dots, 3, 1)$  whose weight equals  $(-a)^k$ . This completes the proof.  $\square$

We should note that the involution of Bessenrodt and Pak [7] also preserves the number of odd parts of  $\lambda \in P_{do}(n)$ . Thus it implies a combinatorial proof of Theorem 4.2 as well.

We also note that Theorem 4.2 can be translated back to the following identity:

$$\sum_{n=1}^{\infty} -aq^{2n-1} (q^{2n}; q^2)_{\infty} (aq^{2n+1}; q^2)_{\infty} = \sum_{k=1}^{\infty} (-a)^k q^{k^2}, \quad (4.5)$$

which takes a different form compared with the identity (1.6). Nevertheless, as shown by Alladi [2], (4.5) can be deduced from (1.6).

## 5. A partition theorem derived from Andrews' identity

As we have seen in the previous section, Theorem 4.2 is a direct translation of the identity (4.5) rather than the identity (1.6). In this section, we derive a partition theorem directly from the identity (1.6) which can be proved with the aid of our involution  $\psi$ . Let  $Q$  denote the set of partitions into distinct nonnegative parts with the smallest part being even, and let  $Q(n)$  denote such partitions of  $n$  in  $Q$ . For a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in Q$ , define the weight of  $\lambda$  by

$$\omega_e(\lambda) = (-1)^l a^{\ell_o(\lambda)}. \quad (5.1)$$

Then we have the following partition identity.

**Theorem 5.1.** We have

$$\sum_{\lambda \in Q(n)} \omega_e(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \quad (5.2)$$

**Proof.** Let  $\lambda$  be a partition in  $Q(n)$ . Let  $s(\lambda)$  denote the smallest part of  $\lambda$ , and let  $ss(\lambda)$  denote the second small part of  $\lambda$ . Define an involution  $\psi$  by the following procedure. Three cases are considered.

- (i) Assume that  $s(\lambda) = 0$  and  $ss(\lambda)$  is even. Delete the part 0 in  $\lambda$  and denote the resulting partition by  $\lambda^*$ . It can be seen that  $\lambda^* \in Q(n)$ .
- (ii) Assume that  $s(\lambda) \neq 0$ . Add 0 to  $\lambda$  as a new part. Denote the resulting partition by  $\lambda^*$ . Then we have  $\lambda^* \in Q(n)$ .

- (iii) Assume that  $s(\lambda) = 0$  and  $ss(\lambda)$  is odd. In this case,  $\lambda$  can be considered as a partition in  $P_{do}(n)$  by disregarding the part 0 so that we can apply  $\Psi$  to  $\lambda$ .

According to the above construction,  $\psi$  is a sign-reversing and weight-preserving involution for which the partition  $\lambda = (2k - 1, 2k - 3, \dots, 3, 1, 0) \in Q(n)$  is defined as the fixed point for  $n = k^2$ . This completes the proof.  $\square$

For example, when  $n = 10$ , there are fourteen partitions in  $Q(10)$ :

$$\begin{array}{cccc} 10, & 8+2, & 6+4, & 5+3+2, \\ 10+0, & 8+2+0, & 6+4+0, & 5+3+2+0, \\ 9+1+0, & 7+3+0, & 4+3+2+1+0, & \\ 7+2+1+0, & 6+3+1+0, & 5+4+1+0. & \end{array}$$

In this example, the involution  $\psi$  gives the following correspondence

$$\begin{array}{l} 10 \leftrightarrow 10+0, \quad 8+2 \leftrightarrow 8+2+0, \quad 6+4 \leftrightarrow 6+4+0, \quad 5+3+2 \leftrightarrow 5+3+2+0, \\ 9+1+0 \leftrightarrow 7+2+1+0, \quad 7+3+0 \leftrightarrow 6+3+1+0, \quad 4+3+2+1+0 \leftrightarrow 5+4+1+0. \end{array}$$

The weights of partitions in  $Q(10)$  are listed below, and it can be seen that  $\psi$  is indeed weight-preserving and sign-reversing,

$$\begin{array}{l} \omega_e(10) = 1, \quad \omega_e(8+2) = -1, \quad \omega_e(6+4) = -1, \quad \omega_e(5+3+2) = a, \\ \omega_e(9+1+0) = a^2, \quad \omega_e(7+3+0) = a^2, \quad \omega_e(4+3+2+1+0) = a^2, \\ \omega_e(10+0) = -1, \quad \omega_e(8+2+0) = 1, \quad \omega_e(6+4+0) = 1, \quad \omega_e(5+3+2+0) = -a, \\ \omega_e(7+2+1+0) = -a^2, \quad \omega_e(6+3+1+0) = -a^2, \quad \omega_e(5+4+1+0) = -a^2. \end{array}$$

To end this section, we remark that the involution of Bessenrodt and Pak can be modified to prove Theorem 5.1.

## 6. Connection to another identity of Andrews

In this section, we consider the special case of Theorem 5.1 when setting  $a = -1$ , that is,

$$\sum_{n=0}^{\infty} q^{2n} (q^{2n+2}; q^2)_{\infty} (-q^{2n+1}; q^2)_{\infty} = \sum_{k=0}^{\infty} q^{k^2}, \quad (6.1)$$

which turns out to be related to a problem proposed by Andrews [3] in 1972, see also, Andrews [4, pp. 156–157]. The original problem of Andrews is stated below.

*A Problem of Andrews.* Let  $q_e(n)$  (resp.  $q_o(n)$ ) denote the number of partitions in  $Q(n)$  that have an even number (resp. odd number) of even parts. Prove that

$$q_o(n) - q_e(n) = \begin{cases} 1, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \quad (6.2)$$

Clearly, the left hand side of (6.1) equals the number of partitions  $\lambda$  in  $Q$  with the sign  $(-1)^{\ell_e(\lambda)-1}$  attached to  $\lambda$ . The sign  $(-1)^{\ell_e(\lambda)-1}$  equals the weight of  $\lambda$  by setting  $a = -1$  in (5.1), namely,

$$\omega_e(\lambda) = (-1)^{l-1} a^{\ell_o(\lambda)} = (-1)^{l-1} (-1)^{\ell_o(\lambda)} = (-1)^{\ell_e(\lambda)-1}.$$

Thus we can apply the above involution  $\psi$  defined in the previous section to give a combinatorial interpretation of the identity (6.2).

When  $a = -1$ , the identity (5.2) can be rewritten as

$$\begin{aligned} \sum_{\lambda \in Q(n)} \omega_e(\lambda) &= \sum_{\lambda \in Q(n)} (-1)^{l-1} (-1)^{\ell_o(\lambda)} = \sum_{\lambda \in Q(n)} (-1)^{\ell_e(\lambda)-1} \\ &= q_o(n) - q_e(n) = \begin{cases} 1, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (6.3)$$

It is clear from (3.1) that the involution  $\psi$  only changes the number of even parts of  $\lambda \in Q(n)$  by 1. Thus the identity (6.2) follows from the involution  $\psi$ .

For example, when  $n = 9$ , the five partitions enumerated by  $q_e(9)$  are

$$8 + 1 + 0, \quad 7 + 2 + 0, \quad 6 + 3 + 0, \quad 5 + 4 + 0, \quad 4 + 3 + 2,$$

and the six partitions enumerated by  $q_o(9)$  are

$$9 + 0, \quad 7 + 2, \quad 6 + 2 + 1 + 0, \quad 5 + 4, \quad 5 + 3 + 1 + 0, \quad 4 + 3 + 2 + 0.$$

Under the involution  $\psi$ , the partitions are paired as follows

$$\begin{aligned} 8 + 1 + 0 &\leftrightarrow 9 + 0, & 7 + 2 + 0 &\leftrightarrow 7 + 2, & 6 + 3 + 0 &\leftrightarrow 6 + 2 + 1 + 0, \\ 5 + 4 + 0 &\leftrightarrow 5 + 4, & 4 + 3 + 2 &\leftrightarrow 4 + 3 + 2 + 0. \end{aligned}$$

The partition  $5 + 3 + 1 + 0$  is the fixed point.

## 7. A more general partition theorem

In this section, we present the following weighted form of Andrews' identity (1.8):

$$\begin{aligned} &\sum_{n=0}^{\infty} q^{2mn} (q^{2mn+2m}; q^{2m})_{\infty} (aq^{2mn+1}; q^2)_{\infty} \\ &= 1 + \sum_{k=1}^{\infty} (-a)^k q^{k^2} \prod_{j=1}^k (1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}), \end{aligned} \quad (7.1)$$

which reduces to (1.8) by setting  $a = -1$  and reduces to (1.6) by setting  $m = 1$ . By extending the involution  $\psi$ , we obtain a combinatorial interpretation of the above generalization. Notice that one can also extend the involution of Bessenrodt and Pak to give a combinatorial proof of (7.1). Using the  $q$ -binomial theorem, Ismail and Stanton [10] gave a generalization of identity (7.1) along with a combinatorial interpretation.

Let us introduce some notation. For a positive integer  $m$ , let  $A_{k,m}$  denote the set of partitions into  $k$  distinct nonnegative parts such that all the even parts are multiples of  $2m$  and the smallest part

is even. Let  $A_m = \bigcup_{k=0}^{\infty} A_{k,m}$  and let  $A_m(n)$  be the set of partitions of  $n$  in  $A_m$ . In this notation, the generating function for partitions  $\lambda \in A_m$  equals

$$\sum_{n=0}^{\infty} q^{2mn} (-q^{2mn+2m}; q^{2m})_{\infty} (-q^{2mn+1}; q^2)_{\infty}. \quad (7.2)$$

To give a combinatorial interpretation of the right hand side of (7.1), let  $H_{k,m}$  denote the set of partitions  $\lambda_{k,m}$  such that each part of  $\lambda_{k,m}$  is less than or equal to  $k$  and the multiplicity of each part is an even number less than  $2m$ . Then the generating function for partitions  $\lambda_{k,m}$  in  $H_{k,m}$  equals

$$\prod_{j=1}^k (1 + q^{2j} + q^{4j} + \dots + q^{2(m-1)j}).$$

The factor  $q^{k^2}$  equals the generating function of the triangular partition

$$T_k = (2k-1, 2k-3, \dots, 3, 1).$$

In order to give a combinatorial explanation of the identity (7.1), we shall give another interpretation of the right hand side of (7.1). To this end, let  $B_{k,m}$  denote the set of partitions  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, 0)$  such that each  $\lambda_i$  is odd and  $0 < \lambda_i - \lambda_{i+1} \leq 2m$  for  $1 \leq i \leq k$ . Set

$$B_m = \bigcup_{k=0}^{\infty} B_{k,m},$$

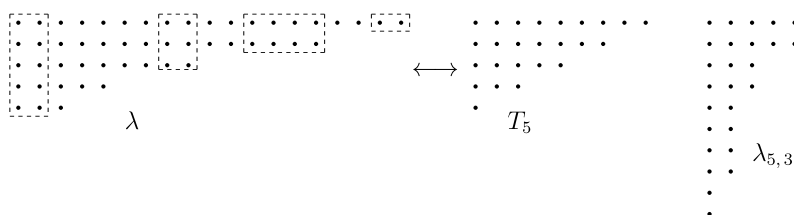
and let  $B_m(n)$  be the set of partitions of  $n$  in  $B_m$ . It is clear that  $B_m(n) \subset A_m(n)$ . For each  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, 0) \in B_{k,m}$ , if we disregard the part 0, then we have the following correspondence.

**Theorem 7.1.** *There exists a bijection between the set  $B_{k,m}$  and the set  $\{T_k\} \times H_{k,m}$ .*

**Proof.** We proceed to construct a bijection from  $B_{k,m}$  to  $\{T_k\} \times H_{k,m}$ . For a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, 0) \in B_{k,m}$ , using the Ferrers diagram and disregarding the part 0, we can generate the triangular partition  $T_k$  and a partition  $\lambda_{k,m} \in H_{k,m}$  by the following procedure. Let  $i$  be the largest integer such that  $\lambda_i - \lambda_{i+1}$  is even, and let  $j = (\lambda_i - \lambda_{i+1})/2$ . Then we have  $i \leq k$  and  $j \leq m$ . We subtract  $2(j-1)$  from each of  $\lambda_1, \lambda_2, \dots, \lambda_i$  and add  $2(j-1)$  parts  $i$  to  $\lambda_{k,m}$ . Repeating this procedure until there does not exist such  $i$ . Finally, the remaining partition is the triangular partition  $T_k$ . It can be seen that  $\lambda_{k,m} \in H_{k,m}$ .

The above construction is reversible. Given the triangular  $T_k$  and a partition  $\lambda_{k,m} \in H_{k,m}$ , let  $\lambda = T_k + \lambda'_{k,m}$ . Then, we have  $\lambda \in B_{k,m}$  with the part 0 added. This completes the proof.  $\square$

Below is an example when  $\lambda = (19, 15, 9, 5, 3, 0) \in B_{5,3}$ .



From Theorem 7.1, we conclude that the generating function for partitions in  $B_m$  equals

$$1 + \sum_{k=1}^{\infty} q^{k^2} \prod_{j=1}^k (1 + q^{2j} + q^{4j} + \dots + q^{2(m-1)j}). \quad (7.3)$$

Using the identities (7.2) and (7.3), we obtain the following number-theoretic interpretation of the identity (7.1) in terms of weighted partitions.

**Theorem 7.2.** Assume that  $m \geq 1$ . For  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k+1}) \in A_m$ , define the weight of  $\lambda$  by

$$\omega_1(\lambda) = (-1)^k a^{\ell_o(\lambda)}. \quad (7.4)$$

Thus, for  $\mu = (\mu_1, \mu_2, \dots, \mu_k, 0) \in B_m$ , the weight of  $\mu$  is given by

$$\omega_2(\mu) = (-a)^k. \quad (7.5)$$

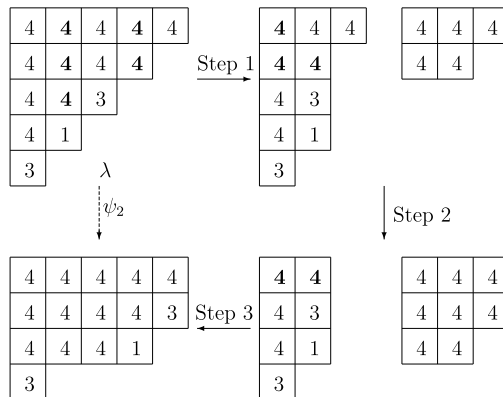
Then the following relation holds

$$\sum_{\lambda \in A_m(n)} \omega_1(\lambda) = \sum_{\mu \in B_m(n)} \omega_2(\mu). \quad (7.6)$$

Since  $A_1 = Q$  and  $B_1$  consists of partitions of the form  $(2k-1, 2k-3, \dots, 3, 1, 0)$ , Theorem 7.2 reduces to Theorem 5.1 when setting  $m = 1$ .

The proof of Theorem 7.2 relies on the notion of  $2m$ -modular diagrams, see [12]. Recall that a  $2m$ -modular diagram of a partition  $\lambda$  is defined to be a Young diagram filled with the numbers  $1, 2, \dots, 2m$  such that the numbers  $1, 2, \dots, 2m-1$  may appear only in the last square of each row and the sum of numbers in the  $i$ -th row equals  $\lambda_i$ .

Let  $P_{do}^m(n)$  denote the set of partitions of  $n$  into distinct parts such that all the even parts are multiples of  $2m$  and the smallest part is odd. Using the  $2m$ -modular diagrams of partitions, we can extend the Franklin type involution  $\psi$  on  $P_{do}(n)$  to  $P_{do}^m(n)$ , and we denote it by  $\psi_m$ . The explicit construction of  $\psi_m$  is analogous to the three steps of the involution  $\psi$  in Section 3, and hence it is omitted. Furthermore, we can extend the involution  $\psi$  on  $Q(n)$  to  $A_m(n)$  with the aid of  $\psi_m$  to give a combinatorial proof of Theorem 7.2, and we denote it by  $\psi_m$ . Since the proof of Theorem 7.2 is similar to that of Theorem 5.1, it is also omitted. Here is an example of the involution  $\psi_m$  for  $m = 2$ . For  $\lambda = (20, 16, 11, 5, 3, 0) \in A_2(55)$ , we have  $\psi_2(\lambda) = (20, 19, 13, 3, 0) \in A_2(55)$ . The following figure is an illustration of the procedure to construct  $\psi_2(\lambda)$ , where the part 0 is disregarded.



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## References

- [1] K. Alladi, A partial theta identity of Ramanujan and its number theoretic interpretation, *Ramanujan J.* 20 (2009) 329–339.
- [2] K. Alladi, A combinatorial study and comparison of partial theta identities of Andrews and Ramanujan, *Ramanujan J.* 23 (2010) 227–241.
- [3] G.E. Andrews, Problem 5865, *Amer. Math. Monthly* 79 (1972) 668.
- [4] G.E. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge, 1998.
- [5] G.E. Andrews, Parity in partition identities, *Ramanujan J.* 23 (2010) 45–90.
- [6] G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook, Part II*, Springer, New York, 2009.
- [7] C. Bessenrodt, I. Pak, Partition congruences by involutions, *European J. Combin.* 25 (2004) 1139–1149.
- [8] B. Berndt, B. Kim, A. Yee, Ramanujan's lost notebook: combinatorial proofs of identities associated with Heine's transformation or partial theta functions, *J. Combin. Theory Ser. A* 117 (2010) 857–973.
- [9] N.J. Fine, Some new results on partitions, *Proc. Natl. Acad. Sci. USA* 34 (1948) 616–618.
- [10] M.E.H. Ismail, D. Stanton, Some combinatorial and analytical identities, *Ann. Comb.*, in press.
- [11] I. Pak, On Fine's partition theorems, Dyson, Andrews, and missed opportunities, *Math. Intelligencer* 25 (2003) 10–16.
- [12] I. Pak, Partition bijections, a survey, *Ramanujan J.* 12 (2006) 5–75.
- [13] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
- [14] A. Stenger, Solution to problem 5865, *Amer. Math. Monthly* 80 (1973) 1147–1148.
- [15] A.J. Yee, Bijective proofs of a theorem of Fine and related partition identities, *Int. J. Number Theory* 5 (2009) 209–218.
- [16] A.J. Yee, Ramanujan's partial theta series and parity in partitions, *Ramanujan J.* 23 (2010) 215–225.